HOMOLOGICAL FINITENESS CONDITIONS FOR GROUPS, MONOIDS AND ALGEBRAS

STEPHEN J. PRIDE

Abstract

Recently Alonso and Hermiller [2] introduced a homological finiteness condition *bi*-FP_n (here called *weak bi*-FP_n) for monoid rings, and Kobayashi and Otto [10] introduced a different property, also called *bi*-FP_n (we adhere to their terminology). From these and other papers we know that: *bi*-$FP_n \Rightarrow$ left and right $FP_n \Rightarrow$ weak *bi*-FP_n; the first implication is not reversible in general; the second implication is reversible for group rings. We show that the second implication is reversible in general, even for arbitrary associative algebras (Theorem $1'$), and we show that the first implication *is* reversible for group rings (Theorem 2). We also show that the all four properties are equivalent for connected graded algebras (Theorem 4). A result on retractions (Theorem $3'$) is proved, and some questions are raised.

1. Introduction

Throughout the paper K will denote a fixed but arbitrary commutative ring, and R, S will denote (not necessarily commutative) rings. All rings will have an identity, and ring homomorphisms will be assumed to preserve the identity.

1.1 Groups and monoids

Let B be a monoid, and let KB be the corresponding monoid ring over K. We have the standard augmentation

$$\varepsilon : KB \to K \quad b \mapsto 1 \quad (b \in B),$$

and we can thus regard K as a left KB-module ${}_BK$ with the KB-action via ε:

$$a.k = \varepsilon(a)k \quad (a \in KB, k \in K).$$

Then B is said to be of type *left*-FP_n (over K) if there is a partial free resolution

$$0 \leftarrow {}_BK \leftarrow P_0 \leftarrow P_1 \leftarrow \ldots \leftarrow P_n \tag{1}$$

where $P_0, P_1, \ldots, P_n$ are *finitely generated* free left KB-modules. Similarly, we can regard K as a right KB-module K_B via ε, and analogously define monoids of type *right*-FP_n by requiring a partial resolution

$$0 \leftarrow K_B \leftarrow P'_0 \leftarrow P'_1 \leftarrow \ldots \leftarrow P'_n \tag{2}$$

by finitely generated free right KB-modules. These two properties are equivalent if there is an involution $*$ on B (that is a mapping ${}^* : B \to B$ satisfying $(bc)^* = c^*b^*, b^{**} = b$ for all $b, c \in B$). In particular, they are equivalent for groups, and more generally inverse monoids (and so in these cases we usually just use the term FP_n). However, in general

Mathematics Subject Classification (MSC): 20M05, 16E40, 20J05

the left and right properties are different. In [6], an example is given of a monoid which is left-FP_∞ (i.e. FP_n for all n) over $\mathbb{Z}$, but not even right-FP_1 over $\mathbb{Z}$ (and vice versa).

We remark that there are examples of groups which are of type FP_n over $\mathbb{Z}$ but not of type FP_{n+1} over $\mathbb{Z}$ for all n [4].

We can also regard K as a (KB, KB)-bimodule ${}_BK_B$ with 2-sided action

$$a.k.a' = \varepsilon(a)k\varepsilon(a') \quad (a, a' \in KB,\ k \in K).$$

We can then define a finiteness condition by requiring that there exists a partial free bi-resolution

$$0 \leftarrow {}_BK_B \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots \leftarrow F_n \tag{3}$$

where $F_0, F_1, \ldots, F_n$ are finitely generated free (KB, KB)-bimodules. This property was introduced in [2], and was called there *bi-FP_n*. However, in this paper we will call it *weak* bi-FP_n, to distinguish it from another property discussed shortly.

As shown in [2],

$$\text{left-}FP_n + \text{right-}FP_n \Longrightarrow \text{weak bi-}FP_n \tag{4}$$

(at least in the case when K is a PID). For, by the Künneth Theorem, the tensor product over K of the free partial resolutions in (1), (2) gives a free partial bi-resolution of ${}_BK \otimes_K K_B \cong {}_BK_B$. (See [2, p.344] for details.)

There is also another "natural" (KB, KB)-bimodule associated with KB, namely KB itself, regarded as a bimodule by left and right multiplication. In [10], the authors defined B to be of type *bi-FP_n* (and we will adhere to their terminology in this paper) if there exists a partial bi-resolution

$$0 \leftarrow KB \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots \leftarrow F_n \tag{5}$$

where $F_0, F_1, \ldots, F_n$ are finitely generated free (KB, KB)-bimodules.

As shown in [10],

$$\text{bi-}FP_n \Longrightarrow \text{left-}FP_n + \text{right-}FP_n. \tag{6}$$

For if we apply $- \otimes_{KB} K$ to (5) then the sequence remains exact and gives a partial resolution of $KB \otimes_{KB} K \cong {}_BK$ by finitely generated free left KB-modules. (See [10, p.338] for details.)

It was proved in [2] that the implication (4) is reversible for *groups*. However, in private correspondence with the second author of [2], it emerged that no example was known to show in general that the implication (4) is not reversible. We will prove that no such example can exist.

Theorem 1 *If a monoid is weak bi-FP_n then it is both left- and right-FP_n.*

As regards the reverse of the implication (6), an example is given in [9] of a monoid which is left- and right-FP_∞ but *is not* bi-FP_3. However, it has been an open question whether (6) is reversible for *groups*. We will show that this is the case.

Theorem 2 *If a group is FP_n then it is bi-FP_n.*

Thus for groups the four properties weak bi-FP_n, left-FP_n, right-FP_n, bi-FP_n all coincide.

Question. Is Theorem 2 true for inverse monoids?

A monoid C is called a *retract* of a monoid B if there are monoid homomorphisms

$$B \underset{\phi}{\overset{\psi}{\rightleftarrows}} C \qquad \psi\phi = \mathrm{id}_C.$$

Theorem 3 *Each of the properties left-FP_n, right-FP_n, bi-FP_n, weak bi-FP_n is closed under retractions.*

(In the case of *groups*, this is proved in [1] for the more general concept of *quasi*-retracts.)

1.2 Algebras

Let A be a K-algebra with an augmentation, that is, a K-algebra epimorphism

$$\varepsilon : A \to K.$$

Then we can regard K as a left A-module ${}_AK$, or a right A-module K_A, or an (A, A)-bimodule ${}_AK_A$ via ε. Also we can regard A as an (A, A)-bimodule ${}_AA_A$ by left and right multiplication. Then we can define A to be *left-FP_n*, *right-FP_n*, *weak bi-FP_n*, *bi-FP_n* if there is a partial free resolution analogous to (1), (2), (3), (5) respectively.

The argument in [2] shows that the implication (4) holds provided K is a PID and A is free (or, more generally, flat) as a K-module. Also, the argument in [10] shows that the implication (6) holds provided A is free (or, more generally, projective) as a K-module. We will show that the *reverse* of (4) holds without restrictions.

Theorem 1′ *If A is weak bi-FP_n, then A is left-FP_n and right-FP_n.*

We remark that Anick [3] showed that if A can be presented as a quotient of a finitely generated free K-algebra by an ideal generated by a finite Gröbner base, then A is left- and right-FP_∞. Recently Kobayashi [8] has improved this to show that such an algebra is bi-FP_∞.

A K-algebra D is a *retract* of A if there are K-algebra homomorphisms

$$A \underset{\theta}{\overset{\kappa}{\rightleftarrows}} D \qquad \kappa\theta = \mathrm{id}_D.$$

Moreover, if D has augmentation

$$\varepsilon_D : D \to K$$

then D is an *augmented retract* if there exist κ, θ as above such that $\varepsilon_D\kappa = \varepsilon$ (and thus $\varepsilon\theta = \varepsilon_D$).

Theorem 3′ (i) *The property bi-FP_n for algebras is closed under retractions.*

(ii) *The properties left-FP_n, right-FP_n, weak bi-FP_n are closed under augmented retractions.*

Theorem 3 follows from this, because if a monoid C is a retract of a monoid B, then the monoid algebra KC is an augmented retract of KB.

We will also consider *connected graded algebras* (definitions will be given in §5).

Theorem 4 *If a connected graded algebra is left-FP_n or right-FP_n then it is bi-FP_n.*

Thus for connected graded algebras, the four properties weak bi-FP_n, left-FP_n, right-FP_n, bi-FP_n all coincide. We remark that the answer to the following seems to be unknown:

Question. For any n, is there a graded algebra of type FP_n but not of type FP_{n+1}?

To prove Theorems $1'$ and $3'$ we first obtain a result concerning what we call *retractive pairs* (see §2). Theorem $3'$ follows directly from this, and Theorem $1'$ follows by working with the enveloping algebra $E = A \otimes_K A^{\mathrm{opp}}$ of A (see §3).

The proof of Theorem 2 is given in §4, and the proof of Theorem 4 is given in §5.

Acknowledgements. I thank Peter Kropholler and Alexandro Olivares for helpful discussions.

2. Retractive pairs and the Property FP_n

The following consequence of the Generalised Schanuel Lemma is useful [5, p 193].

Lemma 1 *Let M be a left R-module. For $n \geq 0$, if*

$$0 \xleftarrow{\partial_{-1}} M \xleftarrow{\partial_0} P_0 \longleftarrow P_1 \ldots \xleftarrow{\partial_{n-1}} P_{n-1}$$

is a partial free resolution of M of length $n-1$, with $P_0, \ldots, P_{n-1}$ finitely generated free modules, then M is of type FP_n if and only if $\ker \partial_{n-1}$ is finitely generated.

Suppose we have ring homomorphisms

$$R \underset{\iota}{\overset{\rho}{\rightleftarrows}} S \qquad \rho\iota = \mathrm{id}_S.$$

ie, ρ is a *retraction* of R onto S, with *section* ι.

A (left) *retractive pair* consists of a left R-module M, a left S-module L, and abelian group homomorphisms

$$M \underset{\alpha^-}{\overset{\alpha^+}{\rightleftarrows}} L \quad \alpha^+\alpha^- = \mathrm{id}_L,$$

where α^+ is an R-module homomorphism (regarding L as an R-module via ρ), and α^- is an S-module homomorphism (regarding M as an S-module via ι). We will denote such a retractive pair by $M\alpha L$.

Retractive pairs form a category, where a *mapping*

$$(\phi, \psi) : M\alpha L \longrightarrow M'\beta L'$$

consists of an R-module homomorphism

$$\phi : M \longrightarrow M'$$

and an S-module homomorphism

$$\psi : L \longrightarrow L',$$

such that the diagram

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, shift right, "\alpha^+"'] & M' \arrow[d, shift right, "\beta^+"'] \\ L \arrow[r, "\psi"] \arrow[u, shift right, "\alpha^-"'] & L' \arrow[u, shift right, "\beta^-"'] \end{tikzcd}$$

commutes. It is then easily checked that $\alpha^+(\mathrm{Ker}\phi) \subseteq \mathrm{Ker}\psi$ and $\alpha^-(\mathrm{Ker}\psi) \subseteq \mathrm{Ker}\phi$, so by restriction, we get the retractive pair $\mathrm{Ker}(\phi, \psi)$:

$$\mathrm{Ker}\phi \underset{\alpha^-}{\overset{\alpha^+}{\rightleftarrows}} \mathrm{Ker}\psi.$$

Similarly, we get the retractive pair $\mathrm{Im}(\phi, \psi)$:

$$\mathrm{Im}\phi \underset{\beta^-}{\overset{\beta^+}{\rightleftarrows}} \mathrm{Im}\psi.$$

Proposition 1 *Let $M\alpha L$ be a retractive pair. If M is of type FP_n then there is a sequence*

$$0 \longleftarrow M\alpha L \overset{(\partial_0,\delta_0)}{\longleftarrow} P_0\beta_0F_0 \overset{(\partial_1,\delta_1)}{\longleftarrow} P_1\beta_1F_1 \longleftarrow \dots \overset{(\partial_n,\delta_n)}{\longleftarrow} P_n\beta_nF_n$$

with P_i, F_i finitely generated free R-modules, S-modules respectively $(0 \leq i \leq n)$, $\mathrm{Im}(\partial_0, \delta_0) =$ $M\alpha L$, *and* $\mathrm{Im}(\partial_{i+1}, \delta_{i+1}) = \mathrm{Ker}(\partial_i, \delta_i)$ $(0 \leq i < n)$.

In particular,

$$0 \longleftarrow L \overset{\delta_0}{\longleftarrow} F_0 \overset{\delta_1}{\longleftarrow} F_1 \longleftarrow \dots \overset{\delta_n}{\longleftarrow} F_n$$

is a partial free resolution of L, so L is of type FP_n.

Proof. Suppose M is of type FP_0 (i.e. finitely generated), and let $\{m_e : e \in \mathbf{e}\}$ be a finite set of R-module generators for M. Then $\{\alpha^+(m_e) : e \in \mathbf{e}\}$ is a set of S-module generators for L. Let P_0 be the free R-module (of rank $2|\mathbf{e}|$)

$$P_0 = (\oplus_{e\in\mathbf{e}}Re) \oplus (\oplus_{e\in\mathbf{e}}Re'),$$

and let F_0 be the free S-module (of rank $|\mathbf{e}|$)

$$F_0 = \oplus_{e\in\mathbf{e}}S\overline{e}.$$

Then

$$P_0 \underset{\beta_0^-}{\overset{\beta_0^+}{\rightleftarrows}} F_0 \qquad \beta_0^+(e) = \overline{e}, \beta_0^+(e') = 0, \beta_0^-(\overline{e}) = e \ (e \in \mathbf{e})$$

is a retractive pair. We have the surjective R-module homomorphism

$$\partial_0 : P_0 \to M, \quad \partial_0(e) = \alpha^-\alpha^+(m_e), \partial_0(e') = m_e - \alpha^-\alpha^+(m_e) \ (e \in \mathbf{e})$$

and the surjective S-module homomorphism

$$\delta_0 : F_0 \to L, \quad \delta_0(\overline{e}) = \alpha^+(m_e) \ (e \in \mathbf{e}),$$

and it is easily checked that (∂_0, δ_0) is a mapping of retractive pairs.

Let $M_1\beta_0 L_1 = \mathrm{Ker}(\partial_0, \delta_0)$. By Lemma 1, if M is if type FP_1, then M_1 is finitely generated. We can then repeat the above procedure to obtain a finitely generated free retractive pair $P_1\beta_1 F_1$ and a surjective map

$$P_1\beta_1 F_1 \to M_1\beta_0 L_1.$$

Composing this with the inclusion of $M_1\beta_0 L_1$ into $P_0\beta_0 F_0$ we obtain a mapping

$$(\partial_1, \delta_1) : P_1\beta_1 F_1 \to P_0\beta_0 F_0.$$

Continuing in this way, after $n+1$ steps we get the required sequence.

3. Bi-Resolutions and Enveloping Algebras

Recall that an (A, A)-bimodule M is an abelian group on which A acts on the left and right, with the condition that: $(am)b = a(mb)$ for all $a, b \in A, m \in M$; $km = mk$ for all $m \in M, k \in K$.

The (A, A)-bimodule $A \otimes_K A$ (with bi-action given by $a.(u \otimes v).b = au \otimes vb$ for all $a, b, u, v \in A$) is free on the generator $1 \otimes 1$. Thus a direct sum of r copies of $A \otimes_K A$ is a *free* (A, A)*-bimodule of rank* r (r may be infinite).

An (A, A)-bimodule M is said to be of *type bi-*FP_n if there is a partial resolution

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots \leftarrow F_n \tag{7}$$

where $F_0, F_1, \ldots, F_n$ are *finitely generated* free (A, A)-bimodules.

It is obvious that Proposition 1 extends to bimodules.

Proof of Theorem 3′. Let A, D be as in the paragraph before the statement of Theorem 3′.

(i) Regarding A, D as (A, A)-, (D, D)-bimodules, respectively, $A \underset{\theta}{\overset{\kappa}{\rightleftarrows}} D$ is a retractive pair.

(ii) Regarding K as a left (respectively, right, bi) A-module, and a left (respectively, right, bi) D-module, $K \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} K$ is a retractive pair.

Recall that for a K-algebra A, there is the *opposite* algebra A^{opp}. This has the same underlying set as A, and the same addition and scalar multiplication. The product of two elements a, b (in that order) in A^{opp} is defined to be the product ba in A. When regarding an element a of A as an element of A^{opp}, we will denote it by a^{opp}.

The *enveloping algebra* of A is the tensor product $E = A \otimes_K A^{opp}$, with multiplication defined by

$$(a \otimes b^{opp})(c \otimes d^{opp}) = ac \otimes b^{opp}d^{opp} = ac \otimes db \quad (a, b, c, d \in A). \tag{8}$$

There is the induced augmentation

$$\varepsilon_E : E \to K \quad \varepsilon_E(a \otimes b^{opp}) = \varepsilon(a)\varepsilon(b) \quad (a, b \in A).$$

If M is an (A, A)-bimodule we can regard it as a left E-module (denoted $\mathcal{E}(M)$) with E-action given by

$$(a \otimes b^{opp})m = amb \quad (a, b \in A, m \in M).$$

Also, if $\phi : M \to M'$ is a bimodule homomorphism, then it can be regarded as a left E-module homomorphism $\mathcal{E}(\phi) : \mathcal{E}(M) \to \mathcal{E}(M')$. Then $\mathcal{E}$ is an (exact) functor. This

functor has an inverse $\mathcal{A}$, where for a left E-module N, $\mathcal{A}(N)$ is N regarded as an (A, A)-bimodule, with left and right actions given by

$$an = (a \otimes 1)n, \; nb = (1 \otimes b^{opp})n \quad (a, b \in A, n \in N).$$

It is easily shown that $\mathcal{E}(A \otimes_K A)$ is E acting on itself by left multiplication. In other words, $\mathcal{E}(A \otimes_K A)$ is a free left E-module of rank 1. Thus, if F is a free (A, A)-bimodule of rank r, then $\mathcal{E}(F)$ is a free left E-module of rank r. Applying $\mathcal{E}$ to a partial resolution as in (7), we thus see that if M is an (A, A)-bimodule of type bi-FP_n, then $\mathcal{E}(M)$ is of type FP_n. By considering the inverse functor $\mathcal{A}$, the converse is also true. Thus we have:

Lemma 2 *An (A, A)-bimodule M is of type bi-FP_n if and only if M, regarded as a left E-module, is of type FP_n.*

Proof of Theorem $1'$**.** Regarding K as an (A, A)-bimodule via ε, $\mathcal{E}(K)$ is easily seen to be K regarded as a left E-module via ε_E. Thus, by Lemma 2, A is weak bi-FP_n if and only if E is left-FP_n. Then, since A is an augmented retract of E under the maps

$$E \underset{\iota}{\overset{\rho}{\rightleftarrows}} A \quad \rho(a \otimes b^{opp}) = \varepsilon(b)a, \; \iota(a) = a \otimes 1 \; (a, b \in A), \; \rho\iota = id_A,$$

if E is left-FP_n then so is A, by Theorem $3'$(ii).

4. Proof of Theorem 2

For a group G we define a functor $- \hat{\otimes} KG$ from the category of left KG-modules to the category of (KG, KG)-bimodules as follows. For M a left KG-module, $M \hat{\otimes} KG$ is the tensor product $M \otimes_K KG$ with bi-KG-action given by

$$g \cdot (m \otimes x) \cdot h = gm \otimes gxh \quad (g, h, x \in G, \; m \in M).$$

For a left KG-module homomorphism $\alpha : M_1 \to M_2$ we define $\hat{\alpha} : M_1 \hat{\otimes} KG \to M_2 \hat{\otimes} KG$ to be

$$\alpha \otimes \mathrm{id}_{KG} : M_1 \otimes_K KG \to M_2 \otimes_K KG,$$

regarded as a (KG, KG)-bimodule homomorphism.

Lemma 3 (i) *$K \hat{\otimes} KG$ is isomorphic to KG (regarded as a (KG, KG)-bimodule by left and right multiplication)*

(ii) If P is a free left KG-module of rank r, then $P \hat{\otimes} KG$ is a free (KG, KG)-bimodule of rank r.

Proof. (i) As an abelian group, $K \hat{\otimes} KG$ is just $K \otimes_K KG$, which is isomorphic to KG by the isomorphism

$$\theta : K \otimes_K KG \to KG \qquad k \otimes x \mapsto kx \; (k \in K, x \in G).$$

It is easily checked that for $g, h, x \in G, \; k \in K$

$$\theta(g \cdot (k \otimes x) \cdot h) \;\; = \;\; g\theta(k \otimes x)h.$$

(ii) Since P is the direct sum of r copies of KG, it suffices to show that $KG \hat{\otimes} KG$ is a free (KG, KG)-module of rank 1. The free (KG, KG)-bimodule of rank 1 is $KG \otimes_K KG$ with action

$$x(g \otimes h)y = xg \otimes hy \quad (x, y, g, h \in G).$$

As a K-module, $KG \otimes_K KG$ is free with basis $g \otimes h$ $(g, h \in G)$. For convenience write $g\hat{\otimes}h$ for $g \otimes h$ when considered as an element of $KG\hat{\otimes}KG$.

Since $KG \otimes_K KG$ is free on $1 \otimes 1$, we get a bi-module homomorphism

$$\beta : KG \otimes_K KG \to KG\hat{\otimes}KG \quad 1 \otimes 1 \mapsto 1\hat{\otimes}1.$$

Thus $\beta(g\otimes h) = \beta(g.(1\otimes 1.h) = g\hat{\otimes}gh$ $(g, h \in G)$. Also, we have a K-module homorphism

$$\alpha : KG\hat{\otimes}KG \to KG \otimes_K KG \quad g\hat{\otimes}h \mapsto g \otimes g^{-1}h \ (g, h \in G).$$

It is easily checked that α and β are mutually inverse (as K-maps), so β is a (KG, KG)-bimodule isomorphism.

To prove Theorem 2, suppose G is left-FP_n over K. Then there is a partial free resolution as in (1). Tensoring by $- \otimes_K KG$ is exact, since KG is free as a K-module, so we obtain the exact sequence

$$0 \leftarrow K \otimes_K KG \leftarrow P_0 \otimes_K KG \leftarrow P_1 \otimes_K KG \leftarrow \cdots \leftarrow P_n \otimes_K KG.$$

Regarding this as a sequence of (KG, KG)-bimodules and using Lemma 3, we obtain an exact sequence

$$0 \leftarrow KG \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n$$

where $F_0, F_1, \cdots, F_n$ are finitely generated free (KG, KG)-bimodules, so G is bi-FP_n.

5. Connected Graded Algebras and Proof of Theorem 4

Suppose K is a field, and that A is a *graded* K-algebra. Thus A is a direct sum $\oplus_{i\geq 0}A_i$ of K-modules such that $A_iA_j \subseteq A_{i+j}$ $(0 \leq i, j)$. Elements of A_i are said to be of *degree* i. In the context of graded algebras, modules will also be graded. Thus a left A-module is a directed sum $M = \oplus_{i\geq 0}M_i$ of K-modules such that $A_iM_j \subseteq M_{i+j}$ $(0 \leq i, j)$. Right modules and bimodule are defined analogously. A module is concentrated in dimension n if $M_i = 0$ for $i \neq n$. We regard K as a graded module concentrated in degree 0. A mapping $\phi : M \to L$ of left modules consists of a family $\phi_i : M_i \to L_i \quad (i \geq 0)$ of K-maps such that $\phi_{i+j}(a_im_j) = a_i\phi_j(m_j)$ $(a_i \in A_i, m_j \in M_j, i, j \geq 0)$.

We will assume that A is *connected*, that is, A_0 has basis the identity 1_A. Then we have the standard augmentation

$$\varepsilon : A \to K \quad 1_A \mapsto 1_K, \quad A_i \to 0 \ (i > 0)$$

with kernel $A^+ = \oplus_{i>0}A_i$.

The opposite algebra A^{opp} is also a connected graded algebra with the same grading, and so $E = A \otimes_K A^{\mathrm{opp}}$ inherits a connected graded algebra structure with grading

$$E_i = \oplus_{p+q=i}A_p \otimes_K A_q$$

and multiplication as in (8).

Remark If M is a graded left A-module then there are two possible definitions of FP_n, according to whether we consider free resolutions

$$0 \longleftarrow M \xleftarrow{\partial_0} P_0 \xleftarrow{\partial_1} P_1 \longleftarrow \ldots \xleftarrow{\partial_i} P_i \longleftarrow \ldots \qquad (9)$$

where the P_i's are free modules which are graded and the ∂_i's are graded maps ("FP_n in the graded sense"), or whether we just consider ungraded resolutions ("FP_n in the ungraded sense"). Clearly, if M is FP_n in the graded sense then it is FP_n in the ungraded sense. The converse is also true. For if M is FP_n in the ungraded sense then

applying $K \otimes_A -$ to an ungraded resolution (9) with $P_0, P_1, \dots, P_n$ finitely generated, we see that $\mathrm{Tor}_i^A(K, M)$ is a finitely generated K-module for $0 \le i \le n$. Now associated with any graded module is a canonical (minimal) graded resolution [7], where the ith term is $A \otimes_K \mathrm{Tor}_i^A(K, M)$, so M is of type FP_n in the graded sense. Similar remarks also hold for right modules, and bimodules.

Proposition 2 *Let M be an (A, A)-bimodule which as a right A-module is free. If the left A-module $M \otimes_A K$ is of type FP_n, then the bimodule M is of type FP_n. (An analogous result holds if we interchange left and right.)*

Taking $M =_A A_A$ in Proposition 2 we obtain Theorem 4.

Proof. Consider the left E-module $\mathcal{E}(M)$. By standard theory [7], there is a unique (up to isomorphism) minimal resolution

$$0 \longleftarrow \mathcal{E}(M) \xleftarrow{\partial_0} P_0 \xleftarrow{\partial_1} P_1 \longleftarrow \dots \xleftarrow{\partial_i} P_i \longleftarrow \dots$$

where P_i is a free E-module, and $\mathrm{Ker}\partial_i \subseteq E^+ P_i$. Applying the functor $\mathcal{A}$, we then get a bi-resolution

$$0 \longleftarrow M \xleftarrow{\overline{\partial}_0} \overline{P}_0 \xleftarrow{\overline{\partial}_1} \overline{P}_1 \longleftarrow \dots \xleftarrow{\overline{\partial}_i} \overline{P}_i \longleftarrow \dots.$$

where $\overline{P}_i$ is a free (A, A)-bimodule and $\mathrm{Ker}\overline{\partial_i} \subseteq \mathcal{A}(E^+ P_i) = A^+ \overline{P}_i + \overline{P}_i A^+$. Note that since a free (A, A)-bimodule is also free as a right A-module, the above sequence is also a free right resolution of M regarded as a right A-module.

Now applying $- \otimes_A K$ we obtain

$$0 \longleftarrow M \otimes_A K \xleftarrow{\overline{\partial}_0 \otimes 1} \overline{P}_0 \otimes_A K \xleftarrow{\overline{\partial}_1 \otimes 1} \overline{P}_1 \otimes_A K \longleftarrow \dots \xleftarrow{\overline{\partial}_i \otimes 1} \overline{P}_i \otimes_A K \longleftarrow \dots.$$

Then for $i \ge 1$

$$\begin{aligned} \tfrac{\mathrm{Ker}\overline{\partial}_i \otimes 1}{\mathrm{Im}\overline{\partial}_{i+1} \otimes 1} &= \mathrm{Tor}_i^A(M, K) \\ &= 0 \ \text{ since } M \text{ is free as a right } A\text{-module.} \end{aligned}$$

So the sequence is exact. Since $\mathrm{Ker}(\overline{\partial}_i \otimes 1) \subseteq A^+(\overline{P}_i \otimes 1)$ the sequence is, in fact, the minimal resolution of $M \otimes_A K$. Thus if $M \otimes_A K$ is of type FP_n, then the left A-modules $\overline{P}_j \otimes_A K$ $(0 \le j \le n)$ are finitely generated. Thus the (A, A)-bimodules

$$(\overline{P}_j \otimes_A K) \otimes_K A \cong \overline{P}_j \quad (0 \le j \le n)$$

are finitely generated, so the bimodule M is of type FP_n.

References

[1] J.M. Alonso, Finiteness conditions on groups and quasi-isometries, J. Pure Appl. Algebra **95** (1994), 121-129.

[2] J.M. Alonso and S.M. Hermiller, Homological finite derivation type, Internat. J. Alg. Comp **13** (2003), 341-359.

[3] D.J. Anick, On the homology of associative algebras, Trans. Amer. Math. Soc. **296** (1987), 641-659.

[4] R. Bieri, Homological dimension of discrete groups, Queen Mary College Mathematics Notes, London, 1976.

[5] K.S. Brown, Cohomology of Groups, Graduate Texts in Mathematics **87**, Springer-Verlag, New York, Heidelberg, Berlin, 1982.

[6] D.E. Cohen, A monoid which is right FP_∞ but not left FP_1, Bull. London Math. Soc. **24** (1992), 340-342.

[7] S. Eilenberg, Homological dimension of syzygies, Annals of Math. **64** (1956), 328-336.

[8] Y. Kobayashi, Gröbner bases of associative algebras and the Hochschild cohomology, Trans. Amer. Math. Soc. **357** (2004), 1095-1124.

[9] Y. Kobayashi and F. Otto, On homotopical and homological finiteness conditions for finitely presented monoids, Internat. J. Alg. Comp **11** (2001), 291-403.

[10] Y. Kobayashi and F. Otto, For finitely presented monoids the homological finiteness conditions FHT and bi-FP_3 coincide, J. Algebra **264** (2003), 327-341.

Address of author:
Department of Mathematics
University of Glasgow
15 University Gardens
Hillhead
Glasgow
G12 8QW

e-mail address:
sjp@maths.gla.ac.uk